MODULE - II

DIFFERENTIAL CALCULUS-I

Course outcome KAS-103T (CO-2)

Apply the concept of limit, continuity and differentiability in the study of Rolle's, Lagrange's, Cauchy mean value theorem and Leibnitz theorem

Lecture 1

Limits of a function

Let f be a function defined in a domain which we take to be an interval, say, I.

We shall study the concept of limit of f at a point 'a' in I. We say $\lim_{x\to a^-} f(x)$ is the expected value of f at x = a given the values of f near to the left of a. This value is called the left hand limit of f at a.

We say $\lim_{x\to a^+} f(x)$ is the expected value of f at x = a given the values of f near to the right of a. This value is called the right hand limit of, f at a.

If the right and left hand limits coincide, we call the common value as the limit of f at x = a and denote it by $\lim_{x\to a} f(x)$.

Some properties of limits Let f and g be two functions such that (Algebra of limits)

1. The limit of a sum (difference) is equal to the sum (difference) of the limits.
2. The limit of product is equal to the product of the limits.
3. The limit of a quotient is equal to the quotient of the limits provided that the limit of the denominator is not equal to zero.

Problems:

Q1.Evaluate $\lim_{x\to 1} \frac{x^3-1}{x^2-1}$. Ans 3/2.

Q2. Evaluate $\lim_{x\to 1} \frac{(1+x)^{1/3}-(1-x)^{1/3}}{x}$. Ans 2/3.

Q3. Evaluate $\lim_{x\to 0}(1+x)^x$. Ans e.

Q4. Evaluate (i) $\lim_{x\to 0} \frac{\sin x}{x}$, (ii).$\lim_{x\to\infty} \frac{\sin x}{x}$. Ans (i) 1, (ii) 0.

Q5. Evaluate $\lim_{x\to 0} \sin\frac{1}{x}$. Ans does not exist

Q6. Let $f(x) = \begin{cases} x, if\ x\ is\ rational \\ -x, if\ x\ is\ irrational \end{cases}$. Show that the limit exists only when $a = 0$.

Continuity:

Definition 1 Suppose f is a real function on a subset of the real numbers and let c be a point in the domain of f. Then f is continuous at c if $\lim_{x\to c} f(x) = f(c)$. otherwise the function is discontinuous at $x = c$.

More elaborately, if the left hand limit, right hand limit and the value of the function at x = c exist and equal to each other, then f is said to be continuous at x = c. Recall that if the right hand and left hand limits at x = c coincide, then we say that the common value is the limit of the function at x = c.

Hence we may also rephrase the definition of continuity as follows: *a function is continuous at x = c if the function is defined at x = c and if the value of the function at x = c equals the limit of the function at x = c. If f is not continuous at c, we say f is discontinuous at c and c is called a point of discontinuity of f.*

Algebra of continuous functions

Theorem 1: Suppose f and g be two real functions continuous at a real number c. Then

(1) f + g is continuous at x = c.

(2) f – g is continuous at x = c.

(3) f . g is continuous at x = c. (4) f /g is continuous at x = c, (provided g (c) ≠ 0).

Problems:

Q1. Discuss the continuity of the function f given by f(x) = | x | at x = 0.

Q2. Discuss the continuity of the function f defined by f (x) = 1/x , x ≠ 0.

Q3. Test the following function for continuity $f(x) = x\sin\frac{1}{x}, x \neq 0, f(0) = 0, at\ x = 0$.

Q4. Examine the function defined below for continuity at $x = a$, $f(x) = \frac{1}{x-a}\text{cosec}, x \neq a, f(x) = 0, at\ x = a$.

Q5. Examine the function defined below for continuity $x = 0$, $f(x) = \frac{\sin^2 ax}{x^2}, x \neq 0, f(0) = 1$.

Lecture 2

Differentiability:

Definitions

Derivative at a point:

Let I, denote the open interval$]a,b[$ in R and let $x_0 \in I$. The a function $f: I \to R$ is said to be differentiable at x_0 iff

$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$ or equivalently $\lim_{h\to 0} \frac{f(x)-f(x_0)}{x-x_0}$ exists finitly and this limit, if it exists finitly, is called the differential coefficient or derivative of f with respect to $x = x_0$. It is denoted by $f'(x_0) or\ Df(x_0)$.

Progressive and regressive derivative:

The progressive derivative of f at $x = x_0$ is given by

$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}, h > 0$. it is also called **right hand differential coefficient** and is denoted by $Rf'(x_0)\ or\ by\ f'(x_0+0)$.

The regressive derivative of f at $x = x_0$ is given by

$\lim_{h\to 0} \frac{f(x_0-h)-f(x_0)}{-h}, h > 0$. it is also called **left hand differential coefficient** and is denoted by $Lf'(x_0)\ or\ by\ f'(x_0-0)$.

It is obvious that that f is differentiable at $x = x_0$, iff $Rf'(x_0)$ and $Lf'(x_0)$ both exists and are equal.

Differentiability in an interval:

Open interval $]a,b[$: A function $f:]a,b[\to R$ is said to be differentiable in $]a,b[$ iff it is differentiable at every point of $]a,b[$.

Closed interval $[a,b]$: A function $f: [a,b] \to R$ is said to be differentiable in $[a,b]$ iff $Rf'(x_0)$ and $Lf'(x_0)$ exists and it is differentiable at every point of $]a,b[$.

Problems:

Q1. Continuity is necessary and sufficient condition for the existence of a finite derivative.

Q2. Prove that the function $f(x) = |x|$ is continuous at $x = 0$, but not differentiable at $x = 0$, where $|x|$ means the numerical or absolute value of x. Also draw the graph.

Q3. Show that the function$f(x) = |x| + |x - 1|$ is not differentiable at $x = 0, x = 1$.

Q4. Prove that the function $f(x) = x \tan^{-1}\frac{1}{x}$ $for\ x \neq 0, f(0) = 0$. is continuous at $x = 0$, but not differentiable at $x = 0$.

Lecture 3

Rolle's Theorem:

Let f be continuous on a closed interval [a, b] and differentiable on the open interval (a, b). If f(a) = f(b), then there is at least one point c in (a, b) where f '(c) = 0.

(The tangent to a graph of f where the derivative vanishes is parallel to x-axis, and so is the line joining the two "end" points (a, f(a)) and (b, f(b)) on the graph. The line that joins to points on a curve -- a function graph in our context -- is often referred to as a *secant*. Thus Rolle's theorem claims the existence of a point at which the tangent to the graph is parallel to the secant, provided the latter is horizontal.)

Problems:

Q1. Discuss the applicability of Rolle's theorem for $f(x) = 2 + (x-1)^{2/3}$ in the interval $[0, 2]$.

Q2. Discuss the applicability of Rolle's theorem for $f(x) = |x|$ in the interval $[-1, 1]$.

Q3. Verify the Rolle's theorem for the following functions

(i) $f(x) = 2x^3 + x^2 - 4x - 2$, (ii) $f(x) = \sin x \; in \; [0, \pi]$

Lagrange's Mean Value Theorem:

Let f be continuous on a closed interval [a, b] and differentiable on the open interval (a, b). Then there is at least one point c in (a, b) where

f '(c) = (f(b) - f(a)) / (b - a).

(The Mean Value Theorem claims the existence of a point at which the tangent is parallel to the secant joining (a, f(a))and (b, f(b)). Rolle's theorem is clearly a particular case of the MVT in which f satisfies an additional condition, f(a) = f(b).)

Problems:

Q1. If $f(x) = (x-1)(x-2)(x-3)$ and $a = 0, b = 4$, find 'c' using Mean value theorem.

Q2. . If $f(x) = x(x-1)(x-2)$ and $a = 0, b = \frac{1}{2}$, find 'c' using Mean value theorem.

Q3. Using Lagrange's Mean value theorem prove that $1 + x < e^x < 1 + xe^x \; \forall x > 0$.

Cauchy's Mean Value Theorem:

Cauchy's mean-value theorem is a generalization of the usual mean-value theorem. It states that if $f(x)$ and $g(x)$ are continuous on the closed interval $[a, b]$, if $g(a) \neq g(b)$, and if both functions are differentiable on the open interval (a, b), then there exists at least one c with $a < c < b$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Problems:

Q1. Verify the Cauchy's Mean theorem for the functions x^2 and x^3in the interval $[1,2]$.

Q2. In Cauchy's mean-value theorem, we write $f(x) = e^x$ and $g(x) = e^{-x}$, show that 'c' is the arithmetic mean between a and b.

Lecture 4

Differential Calculus-I

INTRODUCTION

Calculus is one of the most beautiful intellectual achievements of human being. The mathematical study of change motion, growth or decay is calculus. One of the most important idea of differential calculus is derivative which measures the rate of change of a given function. Concept of derivative is very useful in engineering, science, economics, medicine and computer science.

The first order derivative of y denoted by $\frac{dy}{dx}$, second order derivative, denoted by $\frac{d^2y}{dx^2}$ third order derivative by $\frac{d^3y}{dx^3}$ and so on. Thus by differentiating a function $y = f(x)$, n times, successively, we get the nth order derivative of y denoted by $\frac{d^ny}{dx^n}$ or D^ny or $y_n(x)$. Thus, the process of finding the differential co-efficient of a function again and again is called **Successive Differentiation.**

nth DERIVATIVE OF SOME ELEMENTARY FUNCTIONS

1. Power Function $(ax + b)^m$

Let

$$y = (ax + b)^m$$
$$y_1 = ma\,(ax + b)^{m-1}$$
$$y_2 = m\,(m-1)a^2\,(ax + b)^{m-2}$$

....

....

$$y_n = m(m-1)\,(m-2)\,\ldots\,(m - \overline{n-1})\,a^n\,(ax + b)^{m-n}$$

Case I. When m is positive integer, then

$$y_n = \frac{m\,(m-1)\ldots(m-n+1)(m-n)\ldots3\cdot2\cdot1}{(m-n)\ldots3\cdot2\cdot1}\,a^n(ax + b)^{m-n}$$

$\Rightarrow$

$$\boxed{y_n = \frac{d^n}{dx^n}(ax+b)^m = \frac{\lfloor m}{\lfloor m-n}a^n(ax+b)^{m-n}}$$

Case II. When $m = n =$ +ve integer

$$y_n = \frac{\lfloor n}{\lfloor 0}a^n(ax+b)^0 = \lfloor n\, a^n \Rightarrow \boxed{\frac{d^n}{dx^n}(ax+b)^n = \lfloor n\, a^n}$$

Case III. When $m = -1$, then

$$y = (ax+b)^{-1} = \frac{1}{(ax+b)}$$

$$\therefore \quad y_n = (-1)(-2)(-3) \ldots (-n)\, a^n (ax+b)^{-1-n}$$

$$\Rightarrow \quad \boxed{\frac{d^n}{dx^n}\left\{\frac{1}{ax+b}\right\} = \frac{(-1)^n \lfloor n \, a^n}{(ax+b)^{n+1}}}$$

Case IV. Logarithm case: When $y = \log(ax+b)$, then

$$y_1 = \frac{a}{ax+b}$$

Differentiating $(n-1)$ times, we get

$$y_n = a^n \frac{d^{n-1}}{dx^{n-1}}(ax+b)^{-1}$$

Using case III, we obtain

$$\Rightarrow \quad \boxed{\frac{d^n}{dx^n}\{\log(ax+b)\} = \frac{(-1)^{n-1} \lfloor (n-1)\, a^n}{(ax+b)^n}}$$

2. Exponential Function

(*i*) Consider $y = a^{mx}$

$$y_1 = ma^{mx} . \log_e a$$

$$y_2 = m^2 a^{mx} (\log_e a)^2$$

....

....

$$\boxed{y_n = m^n a^{mx} (\log_e a)^n}$$

(*ii*) Consider $y = e^{mx}$

Putting $a = e$ in above $\boxed{y_n = m^n e^{mx}}$

3. Trigonometric Functions cos (*ax* + *b*) or sin (*ax* + *b*)

Let $y = \cos(ax+b)$, then

$$y_1 = -a \sin(ax+b) = a \cos\left(ax+b+\frac{\pi}{2}\right)$$

$$y_2 = -a^2 \cos(ax+b) = a^2 \cos\left(ax+b+\frac{2\pi}{2}\right)$$

$$y_3 = +a^3 \sin(ax+b) = a^3 \cos\left(ax+b+\frac{3\pi}{2}\right)$$

....

....

$$y_n = \frac{d^n}{dx^n}\cos(ax+b) = a^n \cos\left(ax+b+\frac{n\pi}{2}\right)$$

Similarly,
$$y_n = \frac{d^n}{dx^n}\sin(ax+b) = a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$$

4. Product Functions e^{ax} sin $(bx + c)$ or e^{ax} cos $(bx + c)$

Consider the function $y = e^{ax}\sin(bx+c)$

$$y_1 = e^{ax}.b\cos(bx+c) + ae^{ax}\sin(bx+c)$$
$$= e^{ax}[b\cos(bx+c) + a\sin(bx+c)]$$

To rewrite this in the form of sin, put

$$a = r\cos\phi,\ b = r\sin\phi, \text{ we get}$$
$$y_1 = e^{ax}[r\sin\phi\cos(bx+c) + r\cos\phi\sin(bx+c)]$$
$$y_1 = re^{ax}\sin(bx+c+\phi)$$

Here, $r = \sqrt{a^2+b^2}$ and $\phi = \tan^{-1}(b/a)$

Differentiating again w.r.t. x, we get

$$y_2 = rae^{ax}\sin(bx+c+\phi) + rbe^{ax}\cos(bx+c+\phi)$$

Substituting for a and b, we get

$$y_2 = re^{ax}.\ r\cos\phi\sin(bx+c+\phi) + re^{ax}\ r\sin\phi\cos(bx+c+\phi)$$
$$y_2 = r^2e^{ax}[\cos\phi\sin(bx+c+\phi) + \sin\phi\cos(bx+c+\phi)]$$
$$= r^2 e^{ax}\sin(bx+c+\phi+\phi)$$

$\therefore$
$$y_2 = r^2 e^{ax}\sin(bx+c+2\phi)$$

Similarly,
$$y_3 = r^3e^{ax}\sin(bx+c+3\phi)$$

....

$$y_n = \frac{d^n}{dx^n}e^{ax}\sin(bx+c) = r^ne^{ax}\sin(bx+c+n\phi)$$

In similar way, we obtain

$$y_n = \frac{d^n}{dx^n}e^{ax}\cos(bx+c) = r^ne^{ax}\cos(bx+c+n\phi)$$

Example 1. Find the nth derivative of $\frac{1}{1-5x+6x^2}$

Sol. Let
$$y = \frac{1}{1-5x+6x^2} = \frac{1}{(2x-1)(3x-1)}$$
$$y = \frac{2}{2x-1} - \frac{3}{3x-1} \qquad \text{(By Partial fraction)}$$

$\therefore$
$$y_n = 2\frac{d^n}{dx^n}(2x-1)^{-1} - 3\frac{d^n}{dx^n}(3x-1)^{-1}$$

$$= 2\left[\frac{(-1)^n \lfloor n \, 2^n}{(2x-1)^{n+1}}\right] - 3\left[\frac{(-1)^n \lfloor n \, 3^n}{(3x-1)^{n+1}}\right] \quad \Big| \text{As} \frac{d^n}{dx^n}(ax+b)^{-1} = \frac{(-1)^n \lfloor n \, a^n}{(ax+b)^{n+1}}$$

or $$y_n = (-1)^n \lfloor n \left[\frac{2^{n+1}}{(2x-1)^{n+1}} - \frac{3^{n+1}}{(3x-1)^{n+1}}\right].$$

Example Find the nth derivative of $e^{ax} \cos^2 x \sin x$.

Sol. Let $$y = e^{ax} \cos^2 x \sin x = e^{ax} \frac{(1+\cos 2x)}{2} \sin x$$

$$= \frac{1}{2} e^{ax} \sin x + \frac{1}{2\times 2} e^{ax} (2\cos 2x \sin x)$$

$$= \frac{1}{2} e^{ax} \sin x + \frac{1}{4} e^{ax} \{\sin(3x) - \sin x\}$$

or $$y = \frac{1}{4} e^{ax} \sin x + \frac{1}{4} e^{ax} \sin 3x$$

$\therefore$ $$y_n = \frac{1}{4}\left[r^n e^{ax} \sin(x + n\phi)\right] + \frac{1}{4}\left[r_1{}^n e^{ax} \sin(3x + n\theta)\right].$$

where $r = \sqrt{a^2+1}$; $\tan \phi = 1/a$

and $r_1 = \sqrt{a^2+9}$; $\tan \theta = 3/a$.

Lecture 5

LEIBNITZ'S* THEOREM

Statement. If u and v be any two functions of x, then

$$D^n(u.v) = {}^nc_0\, D^n(u).v + {}^nc_1 D^{n-1}(u).\, D(v) + {}^nc_2\, D^{n-2}(u).D^2(v) + \dots$$
$$+ {}^nc_r D^{n-r}(u).D^r(v) + \dots + {}^nc_n\, u.\, D^n v \quad \dots(i)$$

(U.P.T.U., 2007)

Proof. This theorem will be proved by Mathematical induction.

Now, $\quad D(u.v) = D(u).v + u.D(v) = {}^1c_0\, D(u).v + {}^1c_1\, u.D(v) \quad \dots(ii)$

This shows that the theorem is true for $n = 1$.

Next, let us suppose that the theorem is true for, $n = m$ from (*i*), we have

$$D^m(u.v) = {}^mc_0\, D^m(u).v + {}^mc_1\, D^{m-1}(u)\, D(v) + {}^mc_2\, D^{m-2}(u)\, D^2(v) + \dots + {}^mc_r\, D^{m-r}(u)\, D^r(v) + \dots + {}^mc_m\, u\, D^m(v)$$

Differentiating w.r. to x, we have

$$D^{m+1}(uv) = {}^mc_0\{D^{m+1}(u)\cdot v + D^m(u)\cdot D(v)\} + {}^mc_1\{D^m(u)D(v) + D^{m-1}(u)D^2(v)\}$$
$$+ {}^mc_2\{D^{m-1}(u)D^2(v) + D^{m-2}(u).D^3(v)\} + \dots + {}^mc_r\{D^{m-r+1}(u)D^r v + D^{m-r}(u)D^{r+1}(v)\}$$
$$+ \dots + {}^mc_m\{D(u)\cdot D^m(v) + uD^{m+1}(v)\}$$

But from Algebra we know that ${}^mc_r + {}^mc_{r+1} = {}^{m+1}c_{r+1}$ and ${}^mc_0 = {}^{m+1}c_0 = 1$

$$\therefore\; D^{m+1}(uv) = {}^{m+1}c_0 D^{m+1}(u)\cdot v + ({}^mc_0 + {}^mc_1)D^m(u)\cdot D(v) + ({}^mc_1 + {}^mc_2)D^{m-1}u\cdot D^2v$$
$$+ \dots + ({}^mc_r + {}^mc_{r+1})D^{m-r}(u)\cdot D^{r+1}(v) + \dots + {}^{m+1}c_{m+1}u\cdot D^{m+1}(v)$$

$$\to D^{m+1}(uv) = {}^{m+1}c_0 D^{m+1}(u)\cdot v + {}^{m+1}c_1 D^m(u)\cdot D(v) + {}^{m+1}c_2 D^{m-1}(u)\cdot D^2(v) + \dots$$
$$+ {}^{m-1}c_{r+1} D^{m-r}(u)\cdot D^{r+1}(v) + \dots + {}^{m+1}c_{m+1}u\cdot D^{m+1}(v) \quad \dots(iii)$$

Therefore, the equation (*iii*) shows that the theorem is true for $n = m + 1$ also. But from (2) that the theorem is true for $n = 1$, therefore, the theorem is true for ($n = 1 + 1$) *i.e.*, $n = 2$, and so for $n = 2 + 1 = 3$, and so on. Hence, the theorem is true for all positive integral value of n.

$$\to D^{m+1}(uv) = {}^{m+1}c_0 D^{m+1}(u)\cdot v + {}^{m+1}c_1 D^m(u)\cdot D(v) + {}^{m+1}c_2 D^{m-1}(u)\cdot D^2(v) + \dots$$
$$+ {}^{m+1}c_{r+1} D^{m-r}(u)\cdot D^{r+1}(v) + \dots + {}^{m+1}c_{m+1}u\cdot D^{m+1}(v) \quad \dots(iii)$$

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Example If $y^{1/m} + y^{-1/m} = 2x$, prove that
$(x^2 - 1)\, y_{n+2} + (2n + 1)\, xy_{n+1} + (n^2 - m^2)\, y_n = 0.$

Sol. Given
$$y^{1/m} + \frac{1}{y^{1/m}} = 2x$$

$$\rightarrow \quad y^{2/m} - 2xy^{1/m} + 1 = 0$$

or
$$(y^{1/m})^2 - 2x(y^{1/m}) + 1 = 0$$

$$\Rightarrow \quad z^2 - 2xz + 1 \qquad (y^{1/m} = z)$$

$$\therefore \quad z = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}$$

$$\Rightarrow \quad y^{1/m} = x \pm \sqrt{x^2 - 1} \Rightarrow y = \left[x \pm \sqrt{x^2 - 1}\right]^m \qquad \text{...}(i)$$

Differentiating equation (*i*) w.r.t. x, we get

$$y_1 = m\left[x \pm \sqrt{x^2 - 1}\right]^{m-1}\left[1 \pm \frac{2x}{2\sqrt{x^2 - 1}}\right] = \frac{m\left[x \pm \sqrt{x^2 - 1}\right]^m}{\sqrt{x^2 - 1}}$$

$$\Rightarrow \quad y_1 = \frac{my}{\sqrt{x^2 - 1}} \Rightarrow y_1\sqrt{x^2 - 1} = my$$

or
$$y_1^2\,(x^2 - 1) = m^2y^2 \qquad \text{...}(ii)$$

Differentiating both sides equation (*ii*) w.r.t. x, we obtain

$$2y_1y_2(x^2 - 1) + 2xy_1^2 = 2m^2\, yy_1$$

$$\Rightarrow \quad y_2\,(x^2 - 1) + xy_1 - m^2y = 0$$

Differentiating n times by Leibnitz's theorem w.r.t. x, we get

$$D^n\,(y_2)\cdot(x^2 - 1) + {}^nc_1\, D^{n-1}y_2 \cdot D^2(x^2 - 1) + {}^nc_2\, D^{n-2}\, y_2\, D^2\,(x^2 - 1) + D^n\,(y_1)x + {}^nc_1\, D^{n-1}\,(y_1)\, Dx - m^2y_n = 0$$

$$\Rightarrow \quad y_{n+2}\,(x^2 - 1) + ny_{n+1}\cdot 2x + \frac{n(n-1)}{\underline{|2}}\, y_n\cdot 2 + y_{n+1}\cdot x + ny_n - m^2y_n = 0$$

$$\Rightarrow \quad (x^2 - 1)y_{n+2} + (2n + 1)\, xy_{n+1} + (n^2 - n + n - m^2)y_n = 0$$

$$\Rightarrow \quad (x^2 - 1)\, y_{n+2} + (2n + 1)\, xy_{n+1}\, (n^2 - m^2)\, y_n = 0.$$ **Hence proved.**

Lecture 6

To Find $(y_n)_0$ *i.e.*, *n*th Differential Coefficient of *y*, When $x = 0$

Sometimes we may not be able to find out the *n*th derivative of a given function in a compact form for general value of x but we can find the *n*th derivative for some special value of x generally $x = 0$. The method of procedure will be clear from the following examples:

Example Determine $y_n(0)$ where $y = e^{m.\cos^{-1}x}$.

Sol. We have $y = e^{m.\cos^{-1}x}$ ...(*i*)

Differentiating w.r.t. x, we get

$$y_1 = e^{m\cos^{-1}x}\, m\left(\frac{-1}{\sqrt{1-x^2}}\right) \Rightarrow \sqrt{1-x^2}\,.\,y_1 = -\,me^{m\cos^{-1}x}$$

$$\text{or}\quad \sqrt{1-x^2}\,y_1 = -\,my \Rightarrow (1-x^2)y_1^2 = m^2y^2$$

Differentiating again

$$(1-x^2)\,2y_1\,y_2 - 2xy_1^2 = 2m^2yy_1$$

$$\Rightarrow \quad (1-x^2)y_2 - xy_1 = m^2y \qquad \text{...(ii)}$$

Using Leibnitz's rule differentiating n times w.r.t. x

$$(1 \quad x^2)y_{n+2} \quad 2nxy_{n+1} \quad \frac{2n(n-1)}{\lfloor 2}\,y_n \quad xy_{n+1} \quad ny_n = m^2y_n$$

or

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$$

Putting $x = 0$

$$y_{n-2}(0) - (n^2+m^2)y_n(0) = 0$$

$$\Rightarrow \quad y_{n+2}(0) = (n^2+m^2)y_n(0) \qquad \text{...(iii)}$$

replace n by $(n-2)$

$$y_n(0) = \{(n-2)^2 + m^2\}\,y_{n-2}(0)$$

replace n by $(n-4)$ in equation (*iii*), we get

$$y_{n-2}(0) = \{(n-4)^2 + m^2\}\,y_{n-4}(0)$$

$$\therefore \quad y_n(0) = \{(n-2)^2 + m^2\}\,\{(n-4)^2 + m^2\}\,y_{n-4}(0)$$

Case I. When n is odd:

$$y_n(0) = \{(n-2)^2 + m^2\}\,\{(n-4)^2 + m^2\} \ldots (1^2 + m^2)y_1(0) \qquad \text{...(iv)}$$

[The last term obtain putting $n = 1$ in eqn. (*iii*)]

Now we have $y_1 = -e^{m\cos^{-1}x}\,m\cdot\dfrac{1}{\sqrt{1-x^2}}$

At $x = 0$, $y_1(0) = -me^{\frac{m\pi}{2}}$ ...(*v*) | As $\cos^{-1}0 - \frac{\pi}{2}$

Using (*v*) in (*iv*), we get

$$y_n(0) = -\{(n-2)^2+m^2\}\{(n-4)^2+m^2\}\ldots(1^2+m^2)\,me^{\frac{m\pi}{2}}.$$

Lecture 7

EVOLUTE

Corresponding to each point on a curve we can find the curvature of the curve at that point. Drawing the normal at these points, we can find Centre of Curvature corresponding to each of these points. Since the curvature varies from point to point, centers of curvature also differ. The totality of all such centres of curvature of a given curve will define another curve and this curve is called the evolute of the curve.

"The Locus of centers of curvature of a given curve is called the evolute of that curve."

The locus of the centre of curvature C of a variable point P on a curve is called the evolute of the curve. The curve itself is called involute of the evolute.

Here, for different points on the curve, we get different centre of curvatures. The locus of all these centers of curvature is called as Evolute.

The external curve which satisfies all these centers of curvature is called as Evolute.

Here Evolute is nothing but an curve equation. To find Evolute, the following models exist.

1. If an equation of the curve is given and If we are asked to show / prove L.H.S=R.H.S, Then do as follows.

   First find Centre of Curvature $C(X,Y)$, where $X = x - \frac{y_1[1+y_1^2]}{y_2}$

   $$Y = y + \frac{[1+y_1^2]}{y_2}$$

   And then consider L.H.S: In that directly substitute X in place of$x$ and Yin place of . Similarly for R.H.S. and then show that L.H.S=R.H.S

2. If a curve is given and if we are asked to find the evolute of the given curve, then do as follows:

   First find Centre of curvature $C(X,Y)$ and then re-write as x in terms of X and$y$ in terms of . and then substitute in the given curve, which gives us the required evolute.

3. If a curve is given, which is in parametric form, then first find Centre of curvature, which will be in terms of parameter. Then using these values of X and Y eliminate the parameter, which gives us evolute.

Problems:

Q1. Find the coordinates of centre of curvature at any point of the parabola $y^2 = 4ax$ and also show its evolute is given by $27ay^2 = 4(x-2a)^2$.

Q2. Determine the parametric equations for the evolute of the curve $x = \frac{t^4}{4}, y = \frac{t^5}{5}$.

Ans: $X = -\frac{3}{4}t^4 - t^6, Y = \frac{6}{5}t^5 + t^3$.

Q3. Find the evolute of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Deduce the evolute of a rectangular hyperbola. Ans: $(aX)^{\frac{2}{3}} - (bY)^{\frac{2}{3}} = (a^2+b^2)^{\frac{2}{3}}$, for rectangular hyperbola $(X)^{\frac{2}{3}} - (Y)^{\frac{2}{3}} = (2a)^{\frac{2}{3}}$.

Q4. Find the evolute of the following curves (i) $y = x^3$, (ii) $y = e^x$.

Ans: (i) $X(x) = \frac{x}{2} - \frac{9}{2}x^5, Y(x) = \frac{5}{2}x^3 - \frac{1}{6x}$, (ii) $X(x) = x - 1 - e^{2x}, Y(x) = 2e^{2x} + e^{-x}$.

Lecture 8

Envelopes

A curve which touches each member of a given family of curves is called envelope of that family.

Procedure to find envelope for the given family of curves:

Case 1: Envelope of one parameter Let us consider $y = f(x)$ being the given family of curves.

Step 1: Differentiate w.r.t to the parameter partially, and find the value of the parameter

Step 2: By Substituting the value of parameter in the given family of curves, we get required envelope.

Special Case: If the given equation of curve is quadratic in terms of parameter, then envelope is given by $discriminant = 0$.

Case 2: Envelope of two parameter

Let us consider to be the given family of curves, and a relation connecting these two parameters

Step 1: Obtain one parameter in terms of other parameter from the given relation

Step 2: Substitute in the given equation of curve, so that the problem of two parameter converts to problem of one parameter.

Step 3: Use one parameter technique to obtain envelope for the given family of curve

Problems:

Q1. Find the envelope of the family of straight line $y = mx + \sqrt{a^2m^2 + b^2}$, m is the parameter. Ans: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Q2. Find the envelope of the one parameter family of curves $y = mx + am^p$, where m is the parameter and a, p are constant. Ans: $ap^p y^{p-1} = -x^p . p^{p-1} + (-x)^p$.

Q3. Show that the family of straight lines $2y - 4x + \alpha = 0$ has no envelope, where α being the parameter.

Q4. Determine the envelope of $x\sin t - y\cos t = at$, where t is the parameter.

Q5. Find the envelope of family of straight line $\frac{x}{a} + \frac{y}{b} = 1$, where a, b are two parameters which are connected by the relation $a + b = c$. Ans: $\sqrt{x} + \sqrt{y} = \sqrt{c}$.

Q6. Determine the envelop of two parameter family of parabolas $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$, where a, b are two parameters which are connected by the relation $a + b = c$, where c is a given constant.

Lecture 9

CURVE TRACING

Introduction

It is analytical method in which we draw approximate shape of any curve with the help of symmetry, intercepts, asymptotes, tangents, multiple points, region of existence, sign of the first and second derivatives. In this section, we study tracing of standard and other curves in the cartesian, polar and parametric form.

PROCEDURE FOR TRACING CURVES IN CARTESIAN FORM

The following points should be remembered for tracing of cartesian curves:

Symmetry

(a) Symmetric about x-axis: If all the powers of y occurring in the equation are even then the curve is symmetrical about x-axis.

Example. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ y^2 = 4ax.$

(b) Symmetric about y-axis: If all the powers of x occurring in the equation are even then the curve is symmetrical about y-axis.

Example. $x^2 = 4ay,\ x^4 + y^4 = 4x^2y^2.$

(c) Symmetric about both x- and y-axis: If only even powers of x and y appear in equation then the curve is symmetrical about both axis.

Example. $x^2 + y^2 = a^2.$

(d) Symmetric about origin: If equation remains unchanged when x and y are replaced by $-x$ and $-y$.

Example. $x^5 + y^5 = 5a^2x^2y.$

Remark: Symmetry about both axis is also symmetry about origin but not the converse (due to odd powers).

(e) Symmetric about the line $y = x$: A curve is symmetrical about the line $y = x$, if on interchanging x and y its equation does not change.

Example. $x^3 + y^3 = 3axy.$

(f) Symmetric about $y = -x$: A curve is symmetrical about the line $y = -x$, if the equation of curve remains unchanged by putting $x = -y$ and $y = -x$ in equation.

Example . $x^3 - y^3 = 3axy.$

Regions

(a) Region where the curve exists: It is obtained by solving y in terms of x or vice versa. Real horizontal region is defined by values of x for which y is defined. Real vertical region is defined by values of y for which x is defined.
(b) Region where the curve does not exist: This region is also called imaginary region, in this region y becomes imaginary for values of x or vice versa

Origin and Tangents at the Origin

If there is no constant term in the equation then the curve passes through the origin otherwise not.

If the curve passes through the origin, then the tangents to the curve at the origin are obtained by equating to zero the lowest degree terms.

Example. The curve $a^2y^2 = a^2x^2 - x^4$, lowest degree term $(y^2 - x^2)$ equating to zero gives $y = \pm x$ as the two tangents at the origin.

Intercepts

(a) Intersection point with x- and y-axis: Putting $y = 0$ in the equation we can find points where the curve meets the x–axis. Similarly, putting $x = 0$ in the equation we can find the points where the curve meets y-axis.

(b) Points of intersection: When curve is symmetric about the line $y = \pm x$, the points of intersection are obtained by putting $y = \pm x$ in given equation of curve.

(c) Tangents at other points say (h, k) can be obtained by shifting the origin to these points (h, k) by the substitution $x = x + h$, $y = y + k$ and calculating the tangents at origin in the new xy plane.

Remark. The point where $dy/dx = 0$, the tangent is parallel to x-axis. And the point where $dy/dx = \infty$, the tangent is vertical *i.e.*, parallel to y-axis.

Asymptotes

If there is any asymptotes then find it.

(a) Parallel to x-axis: Equate the coefficient of the highest degree term of x to zero, if it is not constant.

(b) Parallel to y-axis: Equate the coefficient of the highest degree term of y to zero, if it is not constant.

Example. $x^2y - y - x = 0$ highest power coefficient of x *i.e.*, $x^2 = y$

Thus asymptote parallel to x-axis is $y = 0$

Similarly asymptote parallel to y-axis are $x^2 - 1 = 0 \Rightarrow x = \pm 1$.

(c) Oblique asymptotes (not parallel to x-axis and y-axis): The asymptotes are given by

$$y = mx + c, \text{ where } m = \lim_{x \to \infty} \left(\frac{y}{x}\right) \text{ and } c = \lim_{x \to \infty} (y - mx).$$

Sign of First Derivatives dy/dx $(a \le x \le b)$

(a) $\frac{dy}{dx} > 0$, then curve is increasing in $[a, b]$.

(b) $\frac{dy}{dx} < 0$, then curve is decreasing in $[a, b]$.

(c) If $\frac{dy}{dx} = 0$, then the point is stationary point where maxima and minima can occur.

Sign of Second Derivative $\frac{d^2y}{dx^2}$ $(a \le x \le b)$

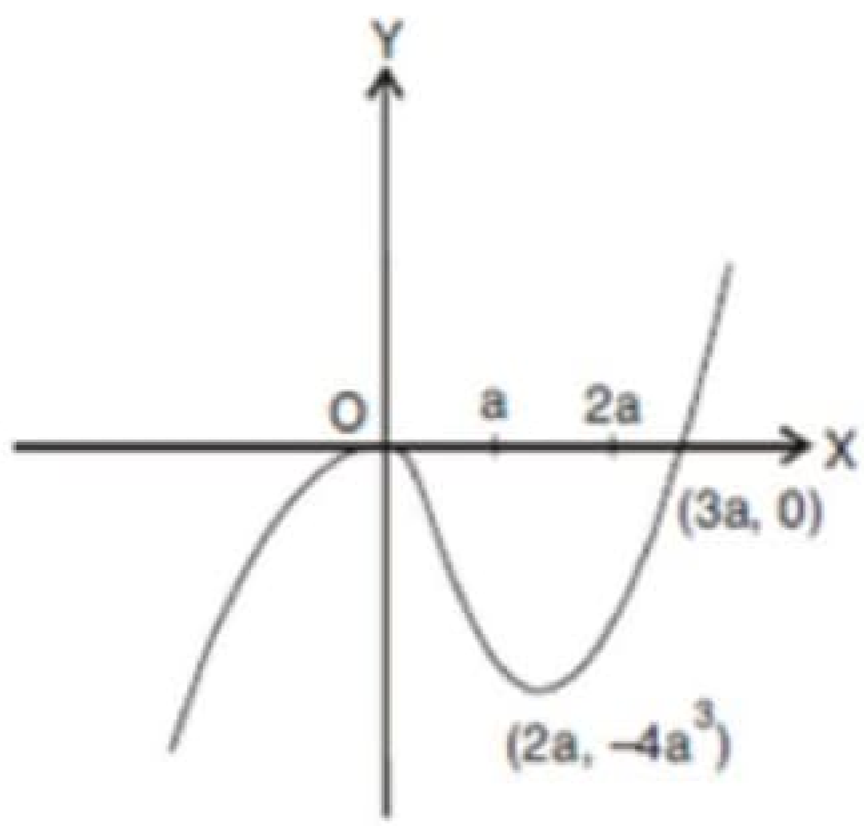
(a) $\frac{d^2y}{dx^2} > 0$, then curve is convex or concave upward (holds water).

(b) $\frac{d^2y}{dx^2} < 0$, then the curve is concave or concave downward (spills water).

Point of Inflexion

A point where $d^2y/dx^2 = 0$ is called an inflexion point where the curve changes the direction of concavity from downward to upward or vice versa.

Example 1. Trace the curve $y = x^3 - 3ax^2$.



Sol. 1. Symmetry: Here the equation of curve do not hold any condition of symmetry. So there is no symmetry.

2. Origin: Since there is no constant add in equation so the curve passes through the origin. The equation of tangent at origin is $y = 0$ *i.e.*, x-axis (lowest degree term).

3. Intercepts: Putting $y = 0$ in given equation, we get

$$x^3 - 3ax^2 = 0 \Rightarrow x = 0, 3a$$

Thus, the curve cross x-axis at $(0, 0)$ and $(3a, 0)$.

4. There is no asymptotes.

5. $\frac{dy}{dx} = 3x^2 - 6ax.$

For stationary point $\frac{dy}{dx} = 0 = 3x^2 - 6ax = 0 \Rightarrow x = 0, 2a.$

6. $\frac{d^2y}{dx^2} = 6x - 6a$, $\left(\frac{d^2y}{dx^2}\right)_{x=0} = -6a < 0$ (concave) and $y_{max} = 0$ and $\left(\frac{d^2y}{dx^2}\right)_{x=2a} = 12a - 6a$ $= 6a > 0$ (convax) and $y_{min.} = -4a^3$.

7. Inflexion point: $\frac{d^2y}{dx^2} = 0 \Rightarrow 6x - 6a = 0 \Rightarrow x = 0, a.$

8. Region: $-\infty < x < \infty$ since y is defined for all x.

Lecture 10

POLAR CURVES

The general form (explicit) of polar curve is $r = f(\theta)$ or $\theta = f(r)$ and the implicit form is $F(r, \theta) = 0$.

Procedure:

1. **Symmetry:** (*a*) If we replace θ by $-\theta$, the equation of the curve remains unchanged then there is symmetry about initial line $\theta = 0$ (usually the positive *x*-axis in cartesian form).

Example. $r = a\,(1 \pm \cos\theta)$.

(*b*) If we replace θ by $\pi - \theta$, the equation of the curve remains unchanged, then there is a symmetry about the line $\theta = \frac{\pi}{2}$ (passing through the pole and $\perp$ to the initial line) which is usually the positive *y*-axis in cartesian).

Example. $r = a \sin 3\theta$.

(*c*) There is a symmetry about the pole (origin) if the equation of the curve remains unchanged by replacing r into $-r$.

Example. $r^2 = a \cos 2\theta$.

(*d*) Curve is symmetric about pole if $f(r, \theta) = f(r, \theta + \pi)$

Example. $r = 4 \tan\theta$.

(*e*) Symmetric about $\theta = \frac{\pi}{4}$ *i.e.*, $(y = x)$, if $f(r, \theta) = f\left(r, \frac{\pi}{2} - \theta\right)$

(*f*) Symmetric about $\theta = \frac{3\pi}{4}$ *i.e.*, $(y = -x)$, if $f(r, \theta) = f\left(r, \frac{3\pi}{2} - \theta\right)$

2. **Pole (origin):** If $r = f(\theta_1) = 0$ for some $\theta = \theta_1$ = constant then curve passes through the pole (origin) and the tangent at the pole (origin) is $\theta = \theta_1$.

Example. $r = a\,(1 + \cos\theta) = 0$, at $\theta = \pi$.

3. **Point of intersection:** Points of intersection of the curve with initial line and line $\theta = \frac{\pi}{2}$ are obtained by putting $\theta = 0$ and $\theta = \frac{\pi}{2}$.

4. **Region:** If r^2 is negative *i.e.*, imaginary for certain values of θ then the curve does not exist for those values of θ.

5. **Asymptote:** If $\lim_{\theta \to \alpha} r = \infty$ then an asymptote to the curve exists and is given by equation

$$r \sin(\theta - \alpha) = f'(\alpha)$$

where α is the solution of $\frac{1}{f(\theta)} = 0$.

6. **Tangent at any point (r, θ):** Tangent at this is obtained from $\tan\phi = \frac{r\,d\theta}{dr}$, where ϕ is the angle between radius vector and the tangent.

7. **Plotting of points:** Solve the equation for r and consider how r varies as θ varies from 0 to ∞ or 0 to $-\infty$. The corresponding values of r and θ give a number of points. Plot these points. This is sufficient for tracing of the curve. (Here we should observe those values of θ for which r is zero or attains a minimum or maximum value).

Example Trace the curve $r^2 = a^2 \cos 2\theta$

Sol. 1. Symmetry: Since there is no change in the curve when θ replace by $-\theta$. So the curve is symmetric about initial line.

2. **Pole:** Curve passes through the pole when $r^2 = a^2 \cos 2\theta = 0$

$$i.e., \cos 2\theta = 0 \Rightarrow 2\theta = \pm \frac{\pi}{2} \text{ or } \theta = \pm \frac{\pi}{4}$$

Hence, the straight lines $\theta = \pm \frac{\pi}{4}$ are the tangents at origin to the curve.

3. **Intersection:** Putting $\theta = 0$

$\therefore\ r^2 = a^2 \Rightarrow r = \pm a$ the curve meets initial line to the points $(a, 0)$ and $(-a, \pi)$.

4. As θ varies from 0 to π, r varies as given below:

$\theta = 0$	30	45	90	135	150	180
$r^2 = a^2$	$a^2/2$	0	$-a^2$	0	$a^2/2$	a^2

←imaginary→

5. **Region:** The above data shows that curve does not exist for values of θ which lying between 45° and 135°.

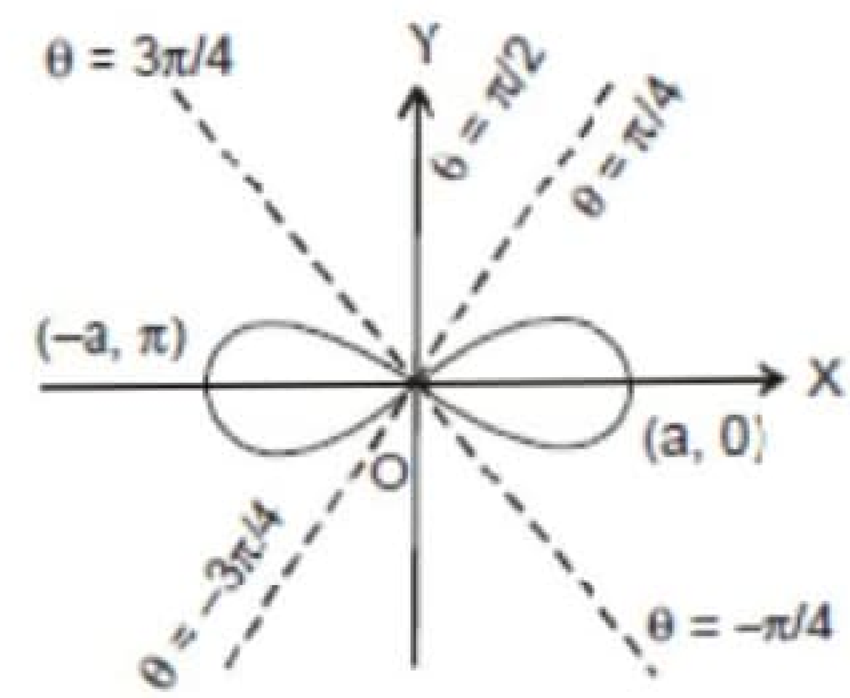


Fig. 1.13